Exam algebraic topology November 2025

Always explain your answers. It is allowed to refer to definitions, lemmas and theorems from the lecture notes but not to other sources. All questions are independent and count equally so make sure you try each of them. Good luck!

1. Suppose $\mathcal{A}$ is an ASC and $\mathcal{B}$ is a subcomplex. Is the complement $\mathcal{A} \setminus \mathcal{B}$ also an ASC? Prove or give a counter example.

2. Give a concrete example of two connected ASC $\mathcal{A}$ and $\mathcal{B}$ with non-isomorphic fundamental groups such that $H_n(\mathcal{A}, \mathbb{F}_p) \cong H_n(\mathcal{B}, \mathbb{F}_p)$ for all n and all primes p. Prove your claims.

3. Suppose you have a connected surface with Euler characteristic 0 and one boundary component. Is its orientable double cover connected?

4. Denote by $\mathcal{A}$ the 2-skeleton of $\mathbb{S}^3$. Show that

$$\dim H_2(\mathcal{A}, \mathbb{Q}) = \dim H_2(\mathcal{A} \cup \mathbb{D}^3) + 1$$

   Recall that $V(\mathbb{S}^3) = \{0, 1, 2, 3, 4\}$ and $V(\mathbb{D}^3) = \{0, 1, 2, 3\}$.

5. Consider a path α starting and ending at point $\mathfrak{b}$ in the connected ASC $\mathcal{B}$. Denote the universal covering of $\mathcal{B}$ by $\mathcal{B}_1$. Describe the endpoint of the lift of α to $\mathcal{B}_1$ that starts at point $(\mathfrak{b}, g)$ where $g \in \pi_1(\mathcal{B}, \mathfrak{b})$.

6. Are the functions $f, g : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ and $g(x) = x^2$ homotopic? If yes, give a homotopy, if no prove no homotopy exists.

7. Define $\mathcal{A} = \langle \{\{k, k+1\} : k \in \mathbb{Z}/7\mathbb{Z}\} \rangle$ and a simplicial map $f : \mathcal{A} \to \mathbb{S}^1$ by $f(0) = 0$, $f(1) = 1$, $f(2) = 2$ and $f(3) = 2$ and $f(4) = 0$ and $f(5) = 1$ and $f(6) = 2$. If

$$\alpha = \overline{[0,1] + [1,2] + [2,3] + [3,4] + [4,5] + [5,6] + [6,0]} \in C_1(\mathcal{A}, \mathbb{Q})$$

   Compute $f_*(\bar{\alpha})$ in terms of the generator of $H_1(\mathbb{S}^1, \mathbb{Q})$.